

Theorem: Let $\mathcal{Q}$ be a quasianalytic class, M be a $\mathcal{Q}$ -manifold, $E \in M$ is a SNC divisor, I a $\mathcal{Q}$ -privileged ideal sheaf. $\forall b \in M$, $\exists$ nbh U of b and a finite family of $\mathcal{Q}$ -morphisms $(\lambda \in \Lambda, |\Lambda| < +\infty)$

$$(*) \quad \sigma_\lambda : (V_\lambda, D_\lambda) \rightarrow (U, E|_U) \in (M, E)$$

such that:

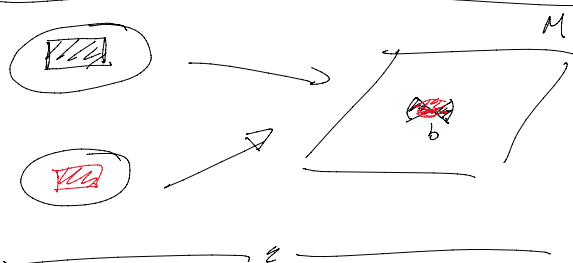
1) σ_λ is the composition of a finite sequence of local blowings-up, (compatible with E)

2) $\{\sigma_\lambda\}$ is a semi-proper covering of U

($\sigma : \sqcup V_\lambda \rightarrow U$, $\sigma|_{V_\lambda} = \sigma_\lambda$, then σ is semi-proper)

3) $\forall \lambda$, $\sigma_\lambda^*(I)$ is monomial.

Def: Let $D \in M$ and $E \in N$ SNC divisors, we say that $\varphi : M \rightarrow N$ is compatible with D and E if $\varphi^{-1}(E)$ is SNC divisor $\subseteq D$. In this case we denote: $\varphi : (M, D) \rightarrow (N, E)$.



Proof: I) Combinatorial techniques

II) Inductive statements

III) Log-Calculus

I) Combinatorial techniques

Let $a \in E$ and fix an E -adapted coordinate system:

$$(u, v) = (u_1, \dots, u_r, v_1, \dots, v_s) \text{ st } E = \left(\prod_{i=1}^r u_i = 0 \right)$$

Def: We say that a privileged ideal sheaf I is

combinatorial if $\forall a \in M$,

$$I_a = (m_1, \dots, m_d) \text{ where } m_k = u^{\alpha_k} = u_1^{\alpha_{k1}} \dots u_r^{\alpha_{kr}} \text{ } \alpha \in \mathbb{N}^r.$$

Rk: Being combinatorial is a local property.

$$\begin{array}{c} a' \\ \swarrow \quad \searrow \\ \text{---} a \text{---} \end{array} \quad \begin{array}{l} I_a = (m_1, \dots, m_d) \\ \Rightarrow I_{a'} = (n_1, \dots, n_{d'}) \end{array}$$

We denote by $\mathcal{D}_E = \text{Der}_M(-\log E)$ the sheaf of $\mathcal{Q}$ -modules of logarithmic derivations.

Locally: $a \in M$

$$\mathcal{D}_{E,a} = \text{Span} \{ u_1 \partial_{u_1}, \dots, u_r \partial_{u_r}, \partial_{v_1}, \dots, \partial_{v_s} \}$$

$$\text{Rk! } u \partial_u (\log u) = \frac{u}{u} = 1,$$

Def: We denote by $\mathcal{D}_E(I)$ the ideal sheaf locally generated by

$$\mathcal{D}_E(I) \cdot \mathcal{Q}_a = \text{Span} \{ X(f) ; X \in \mathcal{D}_{E,a}, f \in I_a \}$$

Ex: $I = (m)$ with $m = u^{\alpha}$, then

$$\mathcal{D}_E(I) + I = (m)$$

$$\begin{cases} u_k \partial_{u_k}(m) = \alpha_k \cdot u^{\alpha} \\ \partial_{v_s}(m) = 0 \end{cases} \Rightarrow \mathcal{D}_E(I) \subseteq I$$

(if $m=1$ and $s=0$, then $\mathcal{D}_E(I) \neq (1)$).

(2) If I is combinatorial, then

$$\mathcal{D}_E(I) + I = I$$

(3) If $I = (v_1^2 + u^{\alpha})$, then:

$$\underline{D_E(I)} \cdot Q_a = \text{Span} \{ X(f) ; X \in D_{E,a} \neq 0 \}$$

stalk at a

Note that

$$\partial_{v_i} (v_i^2 + v^2) = 2v_i \text{ and}$$

$$u_i \partial_{u_i} (v_i^2 + v^2) = 2v_i u_i$$

$$D_E(I) \neq I = (v_i, u_i)$$

(2) If $I = (v_i + v)$, then

$E = (E^{(1)}, \dots, E^{(d)})$, $E^{(j)}$ is an irreducible smooth component and $J \subseteq \{1, \dots, d\}$.

Locally! The center $\mathcal{C}$ is given by (apart from re-ordering $(u_1, \dots, u_r)$):

$$Z(u_1, \dots, u_r), \quad t \leq r.$$

Rk! If I is combinatorial and $\sigma: (\tilde{M}, \tilde{E}) \rightarrow (M, E)$ is a combinatorial blowing-up, then $\sigma^*(I)$ is combinatorial.

Def! We say that a blowing-up

$$\sigma: (\tilde{M}, \tilde{E}) \rightarrow (M, E)$$

is combinatorial if the center $\mathcal{C}$ of E is an intersection $\mathcal{C} = \bigcap_{i \in J} E^{(i)}$ where

Key Lemma! If $\sigma: (\tilde{M}, \tilde{E}) \rightarrow (M, E)$ is a combinatorial blowing-up, then

$$\sigma^*(D_E) = D_{\tilde{E}}$$

Preliminaries:

i) Let $b \in M$ and $a \in \sigma^{-1}(b)$. Let $X \in D_b$.

The pull-back of X at a is the derivation $\sigma^*(X)$ which satisfies the following property

$$\sigma^*(X) \sigma^*(f) = \sigma^*(X(f)) \quad \forall f \in \mathcal{O}_b.$$

ii) $M = \mathbb{R}^2$, $E = \emptyset$, coordinate system (x, y) .

We blow-up the origin and consider x -chart

$$x = u, \quad y = uv$$

We pull back ∂_x :

$$\sigma^*(\partial_x)u = \sigma^*(\partial_x)\sigma^*(x) = 1$$

$$\sigma^*(\partial_x)v = \sigma^*(\partial_x)\sigma^*\left(\frac{y}{x}\right) = -\frac{y}{x^2} = -\frac{v}{u}$$

$$\sigma^*(\partial_x) = \partial_u - \frac{v}{u} \partial_v$$

$\uparrow$
pole

Sketch: Suppose that $\mathcal{C} = Z(u_1, \dots, u_k)$ and consider the u_1 -chart:

$$u_1 = \tilde{u}_1, \quad u_i = \tilde{u}_1 \tilde{u}_i \quad i = 2, \dots, t$$

$$u_j = \tilde{u}_j \quad j = t+1, \dots, r$$

$$v_k = \tilde{v}_k \quad k = 1, \dots, s$$

Claim:

$$\sigma^*(u_1 \partial_{u_1}) = \tilde{u}_1 \partial_{\tilde{u}_1} - \sum_{i=2}^t \tilde{u}_i \partial_{\tilde{u}_i}$$

Consequence! If σ is combinatorial

$$\sigma^*(D_E(I)) = D_{\tilde{E}}(\sigma^*(I))$$

Proposition! Local RS holds for a combinatorial ideal sheaf I under the extra theses:

1) σ is a finite composition of local combinatorial blowings-up.

Sketch: We may find a monomial m such that

$$\sigma^*(u_j \partial_{u_j}) = \tilde{u}_j \partial_{\tilde{u}_j} - \sum_{l=2}^r u_l \partial_{\tilde{u}_l}$$

$$\sigma^*(u_i \partial_{u_i}) = \tilde{u}_i \partial_{\tilde{u}_i} \quad i=2, \dots, r$$

$$\sigma^*(\partial_{v_k}) = \partial_{\tilde{v}_k} \quad k=1, \dots, s$$

□

- Induction in the number of generators d .

It is enough to consider the case $d=2$.

In this case, consider $u=(x,y)$ such that

$$\begin{cases} m_1 = x^\alpha \\ m_2 = y^\beta \end{cases}$$

For a point $b \in M$, we consider:

$$\text{inv}_b(I) = \left(\underbrace{p(b)}_{\substack{\# \text{ exceptional} \\ \text{components at } b}}, \underbrace{\gamma(b)}_{\min\{\alpha, \beta\}}, \underbrace{\delta(b)}_{\max\{\alpha, \beta\}} \right)$$

blowing-up.

Sketch: We may find a monomial m such that

$$I = m(m_1, \dots, m_d)$$

and there is no common term between the $m_1, \dots, m_d$.

Claim: If we chose the combinatorial center "well", this invariant decreases at every point $a \in \sigma^{-1}(b)$.

We treat the case $r=2$. $(x,y) \in \mathbb{R}^2$.

We blow-up with center $Z(x,y)$. Consider the x -chart:

$$\begin{cases} x = u \\ y = w \end{cases}, \quad \gamma(b) = \gamma \quad (u,v) \text{ centered at } a$$

$$m_1^* = u^\alpha = u \cdot u^{\alpha-1}$$

$$m_2^* = u^\beta v^\beta = u^\delta (u^{\beta-\delta} v^\beta)$$

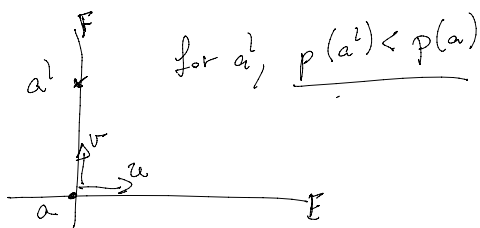
- If $\delta=2$, then $\gamma(a)=0$. Finish.

- If $\delta=\beta$, then:

$$m_1^* = u^\delta [u^{\alpha-\delta}]$$

$$m_2^* = u^\delta [v^\beta] \text{ and}$$

$$\gamma(a) \leq \gamma(b) \text{ and } \delta(a) < \delta(b)$$



□

II) Inductive statements

We prove the result for $I = (\emptyset)$.

(the general case is "harder" notationally-wise).

Notation: $m = \dim M$, we fix $a_0 \in M$.

$$\sqcup V_2 = V_1 \quad \sigma_1: V_1 \rightarrow M \text{ st } \sigma_1|_{V_2} = \sigma_x$$

$$\sqcup E_x = E_1$$

$$I_1 = \sigma_1^*(I)$$

Step I (m) A part from a family $\{\sigma_\lambda\}$ satisfying (1') and (2), $\forall a_1 \in \sigma_1^{-1}(a_0)$

- Either $\sigma_1^*(I)$ is monomial at a_1 .

- Or there is a E_1 -adapted coordinate system (u,v,w) such that

$$f^* = u^\delta \left[w^{\tilde{d}} \tilde{f}(u,v,w) + \sum_{i=1}^{d-1} \tilde{a}_i(u,v) w^i \right]$$

invariant if we consider d minimal in respect to all E_1 -adapted coordinate systems. Fix $a_1 \in V_1$

Step II (m) A part from a family σ_2 satisfying (1) and (2), $\forall a_2 \in \sigma_2^{-1}(a_1)$, we may suppose that

$$f^* = u^\delta \left[w^{\tilde{d}} \tilde{f}(u,v,w) + \sum_{i=0}^{d-2} \tilde{a}_i(u,v) w^i \right]$$

where $\tilde{f}(a_2) \neq 0$, either $b_i(a_2) \neq 0$ or $b_i = 0$,

$$f^* = u \left[w \tilde{f}(u, v, w) + \sum_{i=0}^{d-1} a_i(u, v) w^i \right]$$

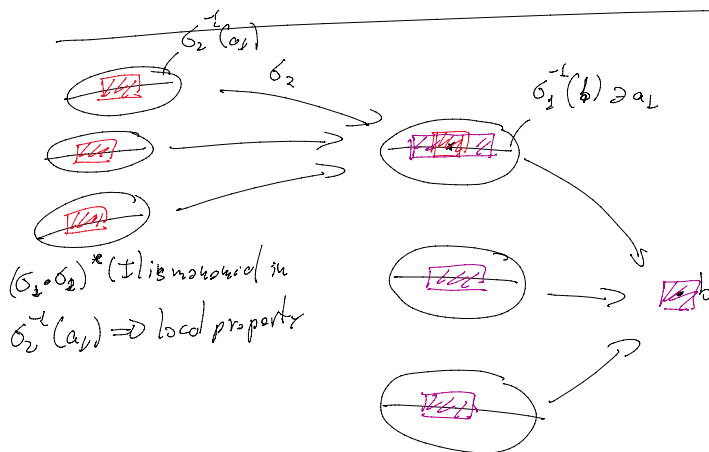
where $\tilde{f}(a_2) \neq 0$, $a_i(a_2) = 0$ and $\tilde{f}, a_i \in Q$.

We set $\nu_{a_1}^{E_1}(I_1) = d$. (it will be an

III (m): Apart from a family σ_3 satisfying (1) and (2), $\forall a_3 \in \sigma_3^{-1}(a_2)$:

$$\nu_{a_3}^{E_3}(I_3) < \nu_{a_2}^{E_2}(I_2)$$

where $\tilde{f}(a_2) \neq 0$, either $b_i(a_2) \neq 0$ or $b_i \equiv 0$, $\tilde{f}, b_i \in Q$, and $\nu_{a_2}^{E_2}(I_2) = \nu_{a_1}^{E_1}$.
Fix $a_2 \in V_2$.



III) Log-Differential Calculus

Def: Let I be a Q -privileged ideal sheaf.

Consider:

$$I = I_0^E \subseteq I_1^E \subseteq \dots \subseteq I_k^E \subseteq \dots$$

$$\text{where } I_{k+1}^E = I_k^E + \mathcal{D}_E(I_k^E)$$

$$\text{Let } I_\infty^E = \sum_{k \in \mathbb{N}} I_k^E$$

Given $a \in M$:

$$\nu_a^E(I) := \min \{ k \in \mathbb{N} \cup \{\infty\}; I_k^E \cdot Q_a = I_\infty^E \cdot Q_a \}$$

Rk: If Q_a is Noetherian then $\nu_a^E(I) < \infty$.

$$E_x(1) \quad I = (f), \text{ where } f = w^d + v_1.$$

Note that $\mathcal{D}_{v_1} f = 1$, so:

$$I + \mathcal{D}_E(I) = (1) \Rightarrow \nu_a^E(I) = 1$$

(2) $I = (f)$ where:

$$f = u \left[w \tilde{f} + \sum_{i=0}^{d-1} a_i(u, v) w^i \right]$$

Claim: $\nu_a^E(I) \leq d$.

$$\text{Note that } \mathcal{D}_w^d f = u \left[\underbrace{d! \tilde{f}}_{\text{unit of } a_1} + \mathcal{O}(1, w) \right]$$

$$\begin{matrix} I_0^E & \subseteq & I_1^E & \subseteq & \dots & \subseteq & I_d^E \\ \cap & & \cap & & & & \cap \\ (u^d) & & (u^d) & & \dots & & (u^d) \end{matrix} \Rightarrow I_d^E = (u^d)$$

Which is invariant by $\mathcal{D}_E$. So $\nu_a^E(I) \leq d$.

Key properties

Note that $\mathcal{O}_W \not\cong u \underbrace{[d\tilde{f} + \mathcal{O}(u, w)]}_{\text{unit of } \mathcal{O}_W}$

$$\Rightarrow u^\lambda \in I_d^E$$

~~Step 1~~ u^λ divides f ; $I_\infty \supset (u^\lambda)$

$$\underline{2}: D_E(u^\lambda) + (u^\lambda) = (u^\lambda)$$

Proof: Direct from the Key-Lemma.

B) Suppose I_∞^E is monomial

i) If $v_a^E(I) = 0 \Rightarrow I$ is monomial

ii) $\forall a, v_a^E(I) < +\infty$

iii) If $v_a^E(I) = d > 0$ and $I = (f)$ then $\exists$ E -adapted coordinate system st

$$f^* = u^\lambda \left[w^d \tilde{f} + \sum_{i=0}^d a_i(u, w) w^i \right]$$

$$\tilde{f}(a) \neq 0, a_i(a) = 0, f, a_i \in \mathbb{Q}$$

Key properties

A) Let $\sigma: (\tilde{M}, \tilde{E}) \rightarrow (M, E)$ be a combinatorial blowing-up and let $J = \sigma^*(I)$. Then

$$J_k^{\tilde{E}} = \sigma^*(I_k^E) \quad \forall k \in \mathbb{N} \cup \{+\infty\}$$

$$\Rightarrow \forall b \in M, \forall a \in \sigma^{-1}(b), v_a^{\tilde{E}}(J) \leq v_b^E(I)$$

Proof of ii)

Let us consider:

$$\mu_a(I) = \max \{k \in \mathbb{N}; I: Q_a \in m_a^k\}$$

$$\text{Then: } \mu_a(I_\infty^E) = \mu_a(u^\lambda) = |\lambda|$$

$$\text{And } (u^\lambda) \supset I_k^E \quad \forall k \in \mathbb{N} \Rightarrow \mu_a(I_k^E) \geq |\lambda|$$

and if $\mu_a(I_k^E) = |\lambda|$ then $I_k^E = I_\infty^E$.

By the definition: $I_\infty^E = \sum_{k \in \mathbb{N}} I_k^E$ we conclude $\exists k$ st.

$$I_{k_0}^E = (u^\lambda) \quad \square$$

Proof of Biii)

$$\hat{f}_a = u^\lambda \left(f_0(w) + \sum_{i \in \mathbb{N}^+} u^i f_i(w) \right)$$

Claim: $f_0(w) \neq 0$ (otherwise, $\mu_a(I_\infty^E) > |\lambda|$)

Apart from a linear change of coordinates, we may suppose that

$$f_0(u, w) = w^{d_0} \tilde{f}_0 \text{ where } \mu_a(f_0) = d_0$$

$$\left(\frac{\partial}{\partial w} f_0(w) \neq 0 \right)$$

And we consider

$$\underline{1}: u^\lambda a_i(u, w) = \frac{\partial^i}{\partial w^i} f(u, w, w) \Big|_{w=0} \in \mathbb{Q} \Rightarrow a_i \in \mathbb{Q}$$

$$f - \sum u^i w^i a_i = u^\lambda w^{d_0} \tilde{f} \in \mathbb{Q} \Rightarrow \tilde{f} \in \mathbb{Q} \quad \square$$

Step I: Consider $I = (f)$. Fix $b \in M$ and take

$$\hat{f}_b = \sum_{i \in \mathbb{N}^+} u^i f_i(w) \text{ and let us consider the toroid } \text{tHtH}$$

$$T = \{u^i; f_i(w) \neq 0\}$$

Rk: T is well-defined everywhere that the coordinate system is well-defined.

Lemma: The following equality holds:

$$T \hat{\Pi} = T^E \cdot \hat{Q}_L$$

$$J_\infty^E = \sigma^*(I_\infty^E)$$

We conclude by Biii.

$$T \cdot \hat{Q}_b = I_\infty^E \cdot \hat{Q}_b$$

We know that T admits a local resolution satisfying (1) and (2). And $\forall a \in \sigma^{-1}(b)$, formally $\sigma^*(T) \cdot \hat{Q}_a = \sigma^*(I_\infty^E) \cdot \hat{Q}_a$.
 monomial $\Rightarrow \sigma^*(I_\infty^E)$ is monomial at a .
 Denote by $J = \sigma^*(I)$. By property A.

We conclude by Bézout.

Proof of Lemma:

Let $\hat{Q}_b = \mathbb{R}[[u, v]]$, is Noetherian. $\exists \alpha_1, \dots, \alpha_d \in \mathbb{N}^r$ and $\beta_1, \dots, \beta_d \in \mathbb{N}^s$ st:

$$\hat{f}_b = \sum_i u^{\alpha_i} v^{\beta_i} f_i(u, v) \text{ where } f_i(0) \neq 0.$$

Let us choose a derivation $X \in D_E$, where

$$X = \sum_{i=1}^r t_i u_i \partial_{u_i} + \sum_{j=1}^s s_j v_j \partial_{v_j}$$

in such a way that:

$$X(u^{\alpha_i} v^{\beta_i}) = k_i \cdot u^{\alpha_i} v^{\beta_i} \text{ and}$$

$\{k_1, \dots, k_d\}$ are distinct.

Note that:

$$X(\hat{f}_b) = \sum_{i=1}^d k_i u^{\alpha_i} v^{\beta_i} (f_i(u, v) + \mathcal{O}(1))$$

$$X^j(\hat{f}_b) = \sum_{i=1}^d k_i^j u^{\alpha_i} v^{\beta_i} (f_i(u, v) + \mathcal{O}(1))$$

$$\begin{bmatrix} X^j(\hat{f}_b) \\ j=0, \dots, d-1 \end{bmatrix} = \underbrace{\begin{bmatrix} k_i^j (f_i(u, v) + \mathcal{O}(1)) \\ \text{at } 0 \end{bmatrix}}_{\text{at } 0} \begin{bmatrix} u^{\alpha_i} v^{\beta_i} \end{bmatrix}$$

$$\begin{bmatrix} k_i^j f_i(0) \end{bmatrix}$$

Vandermonde matrix $\Rightarrow$ invertible!

We conclude that:

$$u^{\alpha_i} v^{\beta_i} \in I_\infty^E \cdot \hat{Q}_b$$

After deriving l times in v

$$u^{\alpha_i} \in I_\infty^E \quad \forall i=1, \dots, d.$$

Therefore $I \subseteq (u^{\alpha_1}, \dots, u^{\alpha_d})$, which is closed

by D_E and $(u^{\alpha_1}, \dots, u^{\alpha_d}) \in I_\infty^E \cdot \hat{Q}_b$

$$\Rightarrow (u^{\alpha_1}, \dots, u^{\alpha_d}) \cdot \hat{Q}_b = I_\infty^E \cdot \hat{Q}_b \quad \square$$

Step II:

$$\hat{f}_b = u^x \left[w^d \tilde{f} + \sum_{i=0}^{d-1} \alpha_i(u, v) w^i \right]$$

Tschinhausser coordinate change:

$$w^{d-1} \tilde{f} = u^x \left(\underbrace{w \cdot \tilde{f}}_{\text{unit}} + \alpha_{d-1}(u, v) (d-1)! \right)$$

It satisfies the Implicit Function Theorem in w .

$\tilde{w} = w - g(u, v)$ so that we get:

$$\tilde{f}_b = u^x \left[w^d \tilde{f} + \sum_{i=0}^{d-1} w^i \alpha_i(u, v) \right]$$

Step III: We blow-up with center $(w=0)$

1. 1. L. 2. F the new exceptional divisor,

$$\tilde{f}_b = v^x \left[w^d \tilde{f} + \sum_{i=0}^{d-1} w^i a_i(u,v) \right]$$

And we apply RS to each $a_i(u,v)$ in order to obtain Step II:

$$\tilde{f}_b = u^x \left[w^d \tilde{f} + \sum_{i=0}^{d-2} w^i u^{\alpha_i} b_i(u,v) \right].$$

$$b_i(0) \neq 0 \text{ or } b_i = 0.$$

By the same argument as in step I:

$$(w^d, u^{\alpha_i} w^i b_i) \subseteq I_{d-1}.$$

Combinatorial

$$\Rightarrow I_{d-1} = (w^d, u^{\alpha_i} w^i b_i)$$

$$\Rightarrow \nu_b^F(I) = d-1.$$

We apply combinatorial resolution to I_{d-1} . We conclude from A and B iii.

Step III: We blow-up with center $(w=v)$ and denote by F the new exceptional divisor,

$$D_F = \{v \partial v_i, \partial_{F_j}, w \partial w\}$$

$$w \partial w(f) = u^x \left[d w^d [\tilde{f} + \partial(f)] + \sum_i i a_i(u,v) \right]$$

$$w \partial w(f) = u^x \left[d w^d [\tilde{f} + \partial(f)] + \sum_i i u^{\alpha_i} b_i w^i \right]$$

$$(w \partial w)^j(f) = u^x \left[d^j w^d [\tilde{f} + \partial(f)] + \sum_{i=0}^{d-2} i^j u^{\alpha_i} w^i b_i \right]$$