Application I : The Uniformization Theorem

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Last course:

Theorem: Let Q be a quasianalytic class, M a Q-manifold, $E \subseteq M$ a SNC divisor and $\mathcal{I}$ a privileged ideal sheaf (of Q). For all relatively compact open set M_0, set $E_0 = E \cap M_0$ and $\mathcal{I}_0 = \mathcal{I} \cdot Q_{M_0}$. There exists a finite sequence of blowings-up:

$$(M_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_1} (M_1, E_1) \longrightarrow (M_0, E_0)$$

where σ_i is E_{i-1}-compatible such that, $\sigma^*(\mathcal{I}_0)$ is monomial, where $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$.

Def: Let $\varphi : N \to M$ be a Q-morphism

- φ is proper if $\forall K \subset\subset M$, $\varphi^{-1}(K)$ is compact
- φ is semi-proper if $\forall K \subset\subset M$, $\exists L \subset\subset N$ s.t. $\varphi(L) = K$.

Lemma I: Let $\varphi : N \to M$ be a semi-proper Q-morphism. $\exists$ a generically local isomorphism Q-morphism $\tau : \tilde{N} \to N$ such that $\tilde{\varphi} = \varphi \circ \tau$ is proper and $\tilde{\varphi}(\tilde{N}) = \varphi(N)$

Idea: The building block for τ was the projection

$$\tilde{\tau} : S^n = \{x_1^2 + \cdots + x_n^2 + t^2 = 1\} \longrightarrow \mathbb{R}^n$$
$$(x, t) \longrightarrow x \quad \square$$

Lemma II: Let $U \subseteq \mathbb{R}^n$ be a connected open set and $f \in Q(U)$ a function s.t. $f \not\equiv 0$. Then $U \setminus Z(f)$ is open dense in U.

Sketch: Let $a \in Z(f) = \{x \in U ;\ f(x) = 0\}$. We know that $\hat{f}_a \not\equiv 0$ (by quasianalyticity, otherwise $f \equiv 0$). So apart from a linear change of coordinates we may apply Malgrange Preparation

Theorem: Let X be a closed Q-set. There exists a proper generically immersive Q-morphism $\varphi : N \to M$ such that $\varphi(N) = X$.

Proof: By Lemma I, it is enough to show the existence of a semi-proper morphism φ (satisfying the other conditions). We may argue locally, so we may suppose

i) There exists a privileged ideal sheaf $\mathcal{I}$ such that $Z(\mathcal{I}) = X$.

ii) There exists a resolution of singularities

$$(M_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, E_1) \xrightarrow{\sigma_1} (M_0, E_0)$$
$$[= (M, \emptyset)]$$

In particular: $\sigma^{-1}(X) \subseteq E_r$.

Induction on $(m, r) = (\dim M, \#\text{blowings-up})$ in respect to the lexicographic order.

Rk: Suppose $X \subseteq E_0$. Recall that $E_0 = \bigcup_{j=1}^{s} E_0^{(j)}$ where $E_0^{(j)}$ are Q-hypersurfaces.

It is enough to apply induction to $(E_0^{(j)}, X \cap E_0^{(j)})$ for all j: $\varphi^{(j)} : N^{(j)} \to M$ s.t. $\varphi^{(j)}(N^{(j)}) = X \cap E_0^{(j)}$ and we conclude easily.

i) If $m = 0$, trivial.

ii) If $(m, 0)$ then $X \subseteq E_0$ and we conclude by the Remark.

iii) If (m, r) for $r > 0$.

Consider the morphisms $\varphi^{(j)} : N^{(j)} \to M$ such that $\varphi^{(j)}(N^{(j)}) = X \cap E_0^{(j)}$ as done in the remark.

Let $\mathcal{C}_1$ be the center of σ_1. Consider $(\mathcal{C}_1, \mathcal{C}_1 \cap X)$. Note that $\dim \mathcal{C}_1 < m$ and apply induction in order to obtain

$$\varphi_1 : N_1 \longrightarrow M \quad \text{s.t.} \quad \varphi_1(N_1) = \mathcal{C}_1 \cap X.$$

After the first blowing-up, let $\mathcal{I}_1 = \sigma_1^*(\mathcal{I})$ and $X_1 = Z(\mathcal{I}_1)$. We apply induction to (M_1, X_1) and obtain

$\tilde{\varphi} : \tilde{N} \longrightarrow M_1$ (a semi-proper generically immersive Q-morphism s.t.

Consider the morphisms such that $\varphi^{(\omega)}(N^{(\omega)}) = X \cap E_0^{(j)}$ as done

$\tilde{\varphi}: \tilde{N} \to M_1$ (a semi-proper generically immersive Q-morphism st.)

$\tilde{\varphi}(\tilde{N}) = X_1$

Consider $\tilde{N} = \bigcup_{\alpha \in \tilde{A}} \tilde{N}_\alpha$ where $\tilde{N}_\alpha$ is connected.

By Lemma II, either

i) $\tilde{\varphi}(\tilde{N}_\alpha) \subseteq E_1$

ii) $\exists\, U_\alpha \subseteq \tilde{N}_\alpha$ open and dense such that $\tilde{\varphi}(U_\alpha) \subseteq X_1 \setminus E_1$.

Let $A = \{\alpha \in \tilde{A} ;\ (ii) \text{ holds}\}$, and let

$N_2 = \bigcup_{\alpha \in A} \tilde{N}_\alpha$ and $\varphi_2 : N_2 \to M$ the composition $\sigma_1 \circ \tilde{\varphi}|_{N_2}$.

We just need to verify that

$$- \varphi_2(N_2) = \overline{X \setminus (E_0 \cup \mathcal{C}_1)}$$

Rk: $\tilde{\varphi}(\tilde{N}) = X_1 \Rightarrow \tilde{\varphi}(N_2) = \overline{X_1 \setminus E_1}$

$$\Rightarrow \sigma_1 \circ \tilde{\varphi}(N_2) = \overline{X \setminus (E_0 \cup \mathcal{C}_1)} \underset{\text{closed}}{\subseteq} X$$

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Consequences:

I) $\dim X := \dim N$.

II)

II) Theorem: Let $X_1 \supseteq \cdots \supseteq X_k \supseteq \cdots$ a sequence of privileged Q-sets of M. Then for all $K \subset\subset M$, $\exists\, k_0 \in \mathbb{N}$ st $X_{k_0} \cap K = X_n \cap K$, $\forall n \geq k_0$.

Sketch: We may suppose that $X_1 \neq M$ and by Lemma II, $\dim X_1 < M$. Use uniformization Theorem

$\varphi : N_1 \to M$ st $\varphi(N_1) = M$.

Pull back: $L = \varphi^{-1}(K)$ (compact)

$Y_k = \varphi^{-1}(X_k)$ $\forall k \in \mathbb{N}$.

Apply induction on the dimension since $\dim N_2 < m$.

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III) Let X be a Q-set, $a \in X$. $\exists$ nbh U of a and $f \in Q(U)$ such that

$$X \cap U = Z(f) \qquad (x_1^2 + x_2^2 \text{ instead of } (x_1 \text{ and } x_2))$$

Exercise.

Def: $X \subseteq M$ is a Q-semianalytic set if $\forall a \in X$, $\exists$ nbh U of a and $f_i \in Q(U)$, $g_{ij} \in Q(U)$, $1 \leq i, j \leq d$ such that

$$X \cap U = \bigcup_{i=1}^{d} \{f_i = 0,\ g_{ij} > 0\}.$$

Def: $X \subseteq M$ is Q-subquasianalytic if $\forall a \in X$, $\exists$ nbh U of a and $Y \subseteq U \times \mathbb{R}^q$, for some q, a Q-semianalytic set such that, if we denote by $\pi : U \times \mathbb{R}^q \to U$ the projection than

- $\pi(Y) = X$
- $\pi|_Y$ is proper.

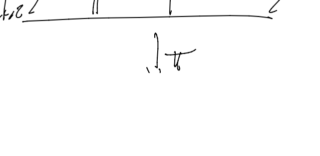
Rk: Proof: - Uniformization for Q-sets

- The set $\{f(x) \geq 0\} \subseteq \mathbb{R}^n$ is the projection of

$$Y = \{f(x) - y^2 = 0\} \subseteq \mathbb{R}^{n+1}$$

- Fiber cutting Lemma.

B - Q-semianalytic



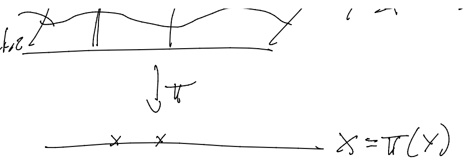
Y Q-semiquasianalytic st $\dim B = \dim X$.

$\downarrow \pi$

· $\pi|_Y$ is proper.

<u>Th:</u> Let X be a closed C-subanalytic set. $\exists$ a proper, generically immersive C-map $\varphi: N \to M$ such that $\varphi(N) = X$.

C-semianalytic



st $\underline{\dim B = \dim X}$.

<u>Ex:</u> (Osgood 1916) Consider $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u,v) = (u, uv, uve^v)$. Show that $\varphi(B_1(0))$ is subanalytic, but not semi-analytic.