MAT336 HOMEWORK 4 (DUE MARCH 09, 2016)

Please write your name, student number and *your section* (0101 or 2001) in every page that you turn in. The section code is important in order to help us upload your grade in Blackboard (each section has a different homepage). The more time we save doing this, the more questions we can correct and the more feedback you will have!

If you turn the assignment in (with most of the questions done) you will get 50%. We will partially mark the questions and the other 50% of the grade depends on the mark obtained on Problems 1 (except (e)), Sec 4.3 L and Sec 5.1 G.

Problems:

For the next problem, you will need the following Theorem (you don't need to prove it):

Theorem 1 (Strong Integral Test)**.** . *Let f be a continuous, positive, decreasing function with domain at least $[1,\infty)$. For every natural number N we define $S_N := \sum_{n=1}^{N} f(n)$ and $I_N = \int_1^{N+1} f(x)dx$. Then there exists a real number C and a sequence (ϵ_n) such that $lim_{n\to\infty}\epsilon_n = 0$ and $S_N = I_N + C + \epsilon_N$ for all N.*

In the next question you will use this result in order to compute the value of $\sum \frac{(-1)^{n+1}}{n}$!

Problem 1. Consider the following series:

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$$
$$B = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$$

Both series contain the exact same terms: they are rearrangements of $\sum \frac{(-1)^{n+1}}{n}$ (actually, $A = \sum \frac{(-1)^{n+1}}{n}$). For every natural number N, we define the N-th harmonic sum H_N and the odd and even harmonic sums as

$$H_N = \sum_{k=1}^{N} \frac{1}{k} \quad E_N = \sum_{k=1}^{N} \frac{1}{2k} \quad O_N = \sum_{k=1}^{N} \frac{1}{2k-1}$$

(a) Find a formula for E_N and O_N in terms of the harmonic sum H (where the index under H may not be N, i.e. it may be H_N, H_{N+1}, H_{2N}, $H_{\frac{1}{2}N}$, etc).

(b) Let $A_N = \sum_{k=1}^{N}(-1)^{k+1}\frac{1}{k}$. Write a formula for A_N that only depends on the harmonic sum H (the formula does not need to be the same for all N).

(c) Use the Strong Integral Test to prove that there exists a real number C and a sequence (ϵ_n) such that:

$$H_N = ln(N+1) + C + \epsilon_n$$

(d) Use (b) and (c) to compute the value of the series A.

(e) Repeat the process in order to compute the value of the series B.

In the following problem, suppose we have fixed some metric space (X, d).

Problem 2. Let $a \in X$ and $\{a_n\}_{n=1}^{\infty}$ a sequence of points in X. Show that $\lim_{n\to\infty} d(a_n, a) = 0$ (as a sequence of real numbers) if and only if $\lim_{n\to\infty} a_n = a$ (as a sequence in (X, d)).
(This should just be an exercise in definitions!)

From the book, you should turn in the following problems:

- Section 3.3: B. Hint (a): Compare with Homework 3; (b) You may want to follow a similar construction as Problem 1 above (i.e. compare the series with the Harmonic one), or use the comparison test in a smart way;
- Section 4.2: A
- Section 4.3: A, B, G, L, M. Hint: 4.3 (b) you may use the topological characterization of continuity; (d) you don't need to do this question, since the argument is similar to (a), but more difficult to write; (B) it is easier to show that the complement is open; L (b) argue by contradiction; M (b) think about the "simplest" closed set possible and intersect it with a dense one.
- Section 5.1: E, G. Hint: (E) is an argument you have seen in Calculus already for the function $\frac{xy^2}{x^2+y^4}$; (G) is a direct consequence of the definition of continuity, or of the topological characterization of it (chose the one you prefer!).
- Section 5.2: A.