MAT336 HOMEWORK 2 (DUE FEB. 3, 2016)

Please write your name, student number and *your section* (0101 or 2001) in every page that you turn in. The section code is important in order to help us upload your grade in Blackboard (each section has a different homepage). The more time we save doing this, the more questions we can correct and the more feedback you will have!

If you turn the assignment in (with most of the questions done) you will get 50%. We will partially mark the questions and the other 50% of the grade depends on the mark obtained on Problems 2, 3 and (Section 2.6 H).

Problems:

Problem 1. Let S be a non-empty subset of $\mathbb{R}$ and M be an upper bound of S. Prove that $M = sup(S)$ if and only if $\forall \epsilon > 0$, there exists $x \in S$ such that:

$$M - \epsilon \leq x \leq M$$

In class we have seen that there exists more than one metric in $\mathbb{R}^n$ (where $n \in \mathbb{N}$). In particular, if $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ are points of $\mathbb{R}^n$ we defined the following norms and (respective) metrics:

$$\begin{aligned} \|\boldsymbol{x}\| &= \sqrt{x_1^2 + \ldots + x_n^2} & d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| \\ \|\boldsymbol{x}\|_\infty &= max\{|x_i|; i = 1, \dots, n\} & d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \end{aligned}$$

In what follows, you will show these metrics are equivalent.

Problem 2. Given any point $\boldsymbol{x} \in \mathbb{R}^n$, show that:

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\| \leq \sqrt{n}\, \|\boldsymbol{x}\|_\infty$$

Problem 3. Let $(a_k)_{k \in \mathbb{N}}$ be a sequence with values in $\mathbb{R}^n$ and L be a point of $\mathbb{R}^n$. Show that $a_n \xrightarrow{d} L$ if and only if $a_n \xrightarrow{d_\infty} L$.

Problem 4. Let (a_n) and (b_n) be sequences of real numbers and L and M be real numbers. Consider the following statement:

- If $(a_n \cdot b_n) \to L \cdot M$ and $a_n \to L$, then $b_n \to M$.

This statement is false!

(a) Find a counterexample.
(b) Find one extra hypothesis concerning *only* L which makes the statement true.
(c) Prove your statement.

From the book, you should turn in the following problems:

- Section 2.3: D (a) - Prove your answer.

- Section 2.9: B, D
- Section 2.4: H
- Section 2.5: D
- Section 2.6: A, H

To answer Sec. 2.9 B, you may use known properties of differentiable functions from calculus.